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ORIGINAL ARTICLE

# Flow over an infinite plate of a viscous fluid with non-integer order derivative without singular kernel

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**Abstract** Exact general solutions for the dynamics of an incompressible viscous fluid with non-integer order derivative without singular kernel are established using the integral transforms. These solutions, which are organized in simple forms in terms of exponential and trigonometric functions, can be conveniently engaged to obtain known solutions from the literature. The control of the new non-integer order derivative on the velocity of the fluid moreover a comparative study with an older model, is analyzed for some flows with practical applications. The non-integer order derivative with non-singular kernel is more appropriate for handling mathematical calculations of obtained solutions. It is also more reliable for numerical computations.
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## 1. Introduction

A discussion on the fractional calculus and its applications at the actual moment, is almost without sense. Fractional calculus is approximately as been around as the standard differential and integral calculus and a list of its applications is too long to be here included. However, it is important to emphasize the fact that fractional derivative generalizations of one-dimensional viscoelastic models have been found to be of great utility in modeling the response linear regime [1] and they are in agreement with the second principle of thermodynamics. Furthermore, as it results from the work of Makris et al. [2], a satisfactory agreement of experimental work was achieved when the non-integer order Maxwell model was used in place of the ordinary one. They also proved that the fractional model has a stronger memory of the recent past than the ordinary model.

During the last decades the fractional calculus has been extensively used and a lot of motion problems have been studied using it [3]. As usually, the governing equations analogous to motions of ordinary fluid models are modified by rehabilitating the integer order time derivatives by the formal left-hand Liouville or Riemann-Liouville differential operators [4–6]. However, these operators as well as the Caputo operator have some drawbacks. Their kernel is singular and the most results that have been recovered using them are expressed in complicated forms involving some generalized functions even for Newtonian fluids [7,8].

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Recently, Caputo and Fabrizio [9] provided a modern definition of non integer order derivative with smooth kernel that works both for temporal and spatial variables. Due to its advantage when the Laplace transform is employed to do problems with initial conditions, this derivative has been already used to solve different real problems [10,11]. A new fractional operator whose kernel is also non-singular was proposed by Atangana and Baleanu [12]. It is based on the Mittag-Leffler function and is useful in thermal science and material sciences. It is worth pointing out that both fractional derivatives, Caputo-Fabrizio and Atangana Baleanu, have all benefits of Riemann-Liouville and Caputo operators. In addition, their kernel is non-singular.

The aim of this note is to use the advantages of one of the modern definitions of non-integer order derivative to obtain exact general solutions for the flow of an incompressible fractional viscous fluid over an infinite plate that is moving in its plane or applies an arbitrary time-dependent tangential stress to the fluid. The obtained results are expressed in simpler forms involving exponential and trigonometric functions and can be conveniently engaged to recover the related solutions for ordinary fluids. Finally, for validation as well as for comparison, three motions with scientific value are considered to obtain as limiting case several results from the literature. Velocity profiles corresponding to two of them are graphically presented and the necessary time to attain the steady-state for oscillating motions is obtained for both models i.e. with Caputo-Fabrizio and Caputo fractional derivative operators.

## 2. Statement of the problem

Consider an infinite plate situated in the $(x, y)$ plane of a fixed Cartesian coordinate system whose positive $y$-axis is in the upward direction, an incompressible viscous fluid is at rest over it at $t = 0$. For the time $t = 0^+$, the plate begins to displace in its plane with a time dependent velocity $Uf(t)$ along the $x$-axis or to apply a shear stress $Sg(t)$ to the fluid in the same direction, the motion is uniform (translation-invariant) in the $x$ and $z$ directions. Here $U$ and $S$ are constants while the non-dimensional functions $f(\dot{s})$ and $g(\dot{s})$ are piecewise continuous and $f(0) = g(0) = 0$. Due to the tangential stress the fluid is also moved and its velocity is of the form of

$$\mathbf{v} = \mathbf{v}(y,t) = (u(y,t), 0, 0). \tag{1}$$

For such motions, the continuity equation is identically verified while the motion and constitutive equations lead to the relevant partial differential equations

$$\frac{\partial \tau(y,t)}{\partial y} = \rho \frac{\partial u(y,t)}{\partial t}, \quad \tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}; \quad y, t > 0, \tag{2}$$

if the body forces as well as pressure gradient along the flow direction are neglected. Here $\rho$ is the fluid density, $\mu$ is its viscosity and $\tau(y, t)$ is the non-zero shear stress.

Relevant initial and boundary conditions corresponding to the two different motions with velocity or shear stress on the boundary are

$$u(y,0) = 0, \quad y > 0; \quad u(0,t) = Uf(t), \quad t > 0, \tag{3}$$

respectively,

$$\tau(y,0) = 0, \quad y > 0; \quad \tau(0,t) = Sg(t), \quad t > 0. \tag{4}$$

Of course, the natural conditions at infinity, namely $u(y, t) \to 0$ and $\tau(y, t) \to 0$ as $y \to \infty$ must be satisfied.

Now to develop the solutions free from the geometry of flow management, we propose the following non-dimensional variables and functions:

$$y^* = \frac{U}{\nu} y, \quad t^* = \frac{U^2}{\nu} t, \quad u^* = \frac{u}{U}, \quad \tau^* = \frac{1}{\rho U^2} \tau,$$
$$f^*(t^*) = f\left(\frac{\nu}{U^2} t^*\right) \tag{5}$$

and

$$y^* = \frac{1}{\nu} \sqrt{\frac{S}{\rho}} y, \quad t^* = \frac{S}{\mu} t, \quad u^* = \sqrt{\frac{\rho}{S}} u, \quad \tau^* = \frac{\tau}{S},$$
$$g^*(t^*) = g\left(\frac{\mu}{S} t^*\right), \tag{6}$$

corresponding to the two different motion problems.

Substituting Eqs. (5) or (6) into Eqs. (2)–(4) and withdrawing the star notation, we find the following non-dimensional initial-boundary value problem:

$$\frac{\partial \tau(y,t)}{\partial y} = \frac{\partial u(y,t)}{\partial t}, \quad \tau(y,t) = \frac{\partial u(y,t)}{\partial y}; \quad y, t > 0, \tag{7}$$

$$u(y,0) = 0, \quad y > 0; \quad u(0,t) = f(t), \quad t > 0;$$
$$u(y,t) \to 0, \text{ as } y \to \infty, \tag{8}$$

respectively

$$\tau(y,0) = 0, \quad y > 0; \quad \tau(0,t) = g(t), \quad t > 0;$$
$$\tau(y,t) \to 0, \text{ as } y \to \infty. \tag{9}$$

Eliminating $\tau(y, t)$ or $u(y, t)$ between Eq. (7), we noticed that the velocity and the shear stress corresponding to both motions satisfy linear partial differential equations of the same style, namely

$$\frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2} \quad \text{or} \quad \frac{\partial \tau(y,t)}{\partial t} = \frac{\partial^2 \tau(y,t)}{\partial y^2}; \quad y, t > 0. \tag{10}$$

Consequently, as the initial and boundary conditions corresponding to both the problems are also identical in form, it is sufficient to solve one of problems and then to use Eq. (7) in order to find the solution of the other problem.

The fractional models corresponding to the two problems are based on the non-integer order partial differential equations [7, Eq. (6.7.44)]:

$$D_t^\alpha u(y,t) = \frac{\partial^2 u(y,t)}{\partial y^2} \quad \text{or} \quad D_t^\alpha \tau(y,t) = \frac{\partial^2 \tau(y,t)}{\partial y^2}; \quad y, t > 0, \tag{11}$$

with the initial and boundary conditions given by Eqs. (8), respectively (9). Here, unlike the previous published papers, the Caputo-Fabrizio derivative operator of order $\alpha$ [9]

$$D_t^\alpha [h(t)] = \frac{1}{1-\alpha} \int_0^t h'(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] ds \text{ for } 0 \leqslant \alpha \leqslant 1, \tag{12}$$

will be used. We firstly solve the fractional differential Eq. $(11)_1$ with the conditions (8) and use the obtained results to develop the solution corresponding to the second problem.

### 3. Solution of the problem

Next to solve the above-mentioned model we employ the integral transform technique. Consequently, applying the Laplace transform [7] to Eq. $(11)_1$ and using (8), we come up with

$$\frac{q}{(1-\alpha)q+\alpha}\bar{u}(y,q)=\frac{\partial^2\bar{u}(y,q)}{\partial y^2};\quad \bar{u}(0,q)=F(q),$$
$$\bar{u}(y,q)\to 0 \text{ as } y\to\infty, \tag{13}$$

where $\bar{u}(y,q)$ and $\overline{F}(q)$ are the Laplace transforms of $u(y,t)$ and $f(t)$ while $q$ is the transform parameter. Further on multiplying Eq. $(13)_1$ by $\sqrt{\frac{2}{\pi}}\sin(y\xi)$ and integrating with respect to $y$ from 0 to $\infty$ by utilizing the conditions $(13)_2$ and $(13)_3$, it results that

$$\bar{u}_s(\xi,q)=F(q)\sqrt{\frac{2}{\pi}}\frac{[(1-\alpha)q+\alpha]\xi}{[1+(1-\alpha)\xi^2]q+\alpha\xi^2}, \tag{14}$$

where $\bar{u}_s(\xi,q)=\sqrt{\frac{2}{\pi}}\int_0^\infty\bar{u}(y,q)\sin(y\xi)d\xi$ is the Fourier sine transform of $\bar{u}(y,q)$.

To present our solution in a suitable form, we write $\bar{u}_s(\xi,q)$ under the form

$$\bar{u}_s(\xi,q)=F(q)\sqrt{\frac{2}{\pi}}\left\{\frac{1}{\xi}\left[1-\frac{1}{1+(1-\alpha)\xi^2}\right]+\frac{\alpha\xi}{[1+(1-\alpha)\xi^2]^2[q+a(\xi)]}\right\}, \tag{15}$$

where $a(\xi)=\frac{\alpha\xi^2}{1+(1-\alpha)\xi^2}$. Using the inverse Laplace transform together with the convolution theorem, to the last relation, we find that

$$u_s(\xi,t)=\frac{f(t)}{\xi}\sqrt{\frac{2}{\pi}}\left[1-\frac{1}{1+(1-\alpha)\xi^2}\right]+\frac{\alpha\xi}{[1+(1-\alpha)\xi^2]^2}\sqrt{\frac{2}{\pi}}\int_0^t f(t-s)e^{-a(\xi)s}ds. \tag{16}$$

By now applying the inverse Fourier sine transform of Eq. (16), we find the velocity field in the simple form

$$u(y,t)=f(t)\left\{1-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)d\xi}{\xi[1+(1-\alpha)\xi^2]}\right\}+\frac{2\alpha}{\pi}\int_0^\infty\frac{\xi\sin(y\xi)}{[1+(1-\alpha)\xi^2]^2}\int_0^t f(t-s)e^{-a(\xi)s}dsd\xi, \tag{17}$$

which satisfies the initial and boundary conditions (8). Introducing Eq. (17) into $(7)_2$ we attain the analogous shear stress, namely

$$\tau(y,t)=-\frac{f(t)}{\sqrt{1-\alpha}}\exp\left(-\frac{y}{\sqrt{1-\alpha}}\right)+\frac{2\alpha}{\pi}\int_0^\infty\frac{\xi^2\cos(y\xi)}{[1+(1-\alpha)\xi^2]^2}\int_0^t f(t-s)e^{-a(\xi)s}dsd\xi. \tag{18}$$

For validation, let us take the limit of Eqs. (17) and (18) for $\alpha\to 1$. Straightforward computation shows that

$$u(y,t)=\frac{y}{2\sqrt{\pi}}\int_0^t\frac{f(t-s)}{s\sqrt{s}}\exp\left(-\frac{y^2}{4s}\right)ds, \tag{19}$$

$$\tau(y,t)=\frac{1}{2\sqrt{\pi}}\int_0^t\frac{f(t-s)}{s\sqrt{s}}\exp\left(-\frac{y^2}{4s}\right)ds-\frac{y^2}{4\sqrt{\pi}}\int_0^t\frac{f(t-s)}{s^2\sqrt{s}}\exp\left(-\frac{y^2}{4s}\right)ds. \tag{20}$$

As expected, the velocity field (19) is identical to that obtained by Fetecau et al. [13, Eq. (31) with $K_{eff}=0$] while the shear stress (20) can be also obtained by introducing Eq. (19) into $(7)_2$.

Finally, it is worth pointing out that once the first problem has been solved, the solutions of the second one can be easily obtained. Indeed, according to the governing Eq. (11) and the boundary conditions (8) and (9), the dimensionless shear stress field corresponding to the motion generated by a plate that exerts a shear stress $Sg(t)$ to the fluid is given by (see Eq. (17)):

$$\tau(y,t)=g(t)\left\{1-\frac{2}{\pi}\int_0^\infty\frac{\sin(y\xi)d\xi}{\xi[1+(1-\alpha)\xi^2]}\right\}+\frac{2\alpha}{\pi}\int_0^\infty\frac{\xi\sin(y\xi)}{[1+(1-\alpha)\xi^2]^2}\int_0^t g(t-s)e^{-a(\xi)s}dsd\xi. \tag{21}$$

In order to determine the associate velocity field, we need the fractional form

$$\frac{\partial\tau(y,t)}{\partial y}=D_t^\alpha u(y,t);\quad y,t>0, \tag{22}$$

of the motion Eq. $(7)_1$. Implementing the Laplace transform to Eq. (22) and taking into account the fact that the fluid is at rest at the moment $t=0$, we find that

$$\bar{u}(y,q)=\frac{(1-\alpha)q+\alpha}{q}\frac{\partial\bar{\tau}(y,q)}{\partial y};\quad y>0, \tag{23}$$

where

$$\frac{\partial\bar{\tau}(y,q)}{\partial y}=-\frac{2G(q)}{\pi}\int_0^\infty\frac{\cos(y\xi)}{1+(1-\alpha)\xi^2}d\xi+\frac{2\alpha}{\pi}\int_0^\infty\frac{\xi^2\cos(y\xi)}{[1+(1-\alpha)\xi^2]^2}\frac{G(q)}{q+a(\xi)}d\xi, \tag{24}$$

is immediately obtained from Eq. (21) and $G(q)$ is the Laplace transform of the function $g(t)$.

Bringing Eq. (24) into (23) and applying the inverse Laplace transform, we obtain after some unconcealed computations that

$$u(y,t)=\frac{2}{\pi}(\alpha-1)g(t)\int_0^\infty\frac{\cos(y\xi)}{1+(1-\alpha)\xi^2}d\xi-\frac{2\alpha}{\pi}\int_0^\infty\times\frac{\cos(y\xi)}{[1+(1-\alpha)\xi^2]^2}\int_0^t g(t-s)e^{-a(\xi)s}dsd\xi. \tag{25}$$

The solutions corresponding to ordinary fluids, namely

$$u(y,t)=-\frac{1}{\sqrt{\pi}}\int_0^t\frac{g(t-s)}{\sqrt{s}}\exp\left(-\frac{y^2}{4s}\right)ds,\quad \tau(y,t)=\frac{y}{2\sqrt{\pi}}\int_0^t\frac{g(t-s)}{s\sqrt{s}}\exp\left(-\frac{y^2}{4s}\right)ds, \tag{26}$$

are immediately obtained from Eqs. (21) and (25) for $\alpha\to 1$. They represent the dimensionless forms of the solutions obtained by Fetecau et al. [14, Eqs. (23) and (24)] by a different technique.

## 4. Numerical results and discussion

With a view to get a little insight of the results that have been obtained, three special cases with engineering applications are considered along with different graphical representations and are presented and discussed. To avoid duplication (because the velocity fields considering to the first motion are identical as form to the shear stresses of the second motion), as well as for comparison, the graphical representations are prepared only for the velocity fields corresponding to the two motions.

### *4.1. Case 1 (constant velocity or constant shear on the boundary)*

Taking $f(t) = H(t)$ and $g(t) = -H(t)$ (see [15]) into Eqs. (17) and (25), where $H(\dot{s})$ is the Heaviside unit step function, we find the dimensionless velocity fields

$$u_v(y,t)=H(t)\left\{1-\frac{2}{\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi\left[1+(1-\alpha)\xi^2\right]}\exp\left(-\frac{\alpha\xi^2 t}{1+(1-\alpha)\xi^2}\right)d\xi\right\}, \tag{27}$$

$$u_s(y,t)=\frac{2}{\pi}H(t)\int_0^\infty\left\{1-\frac{1}{1+(1-\alpha)\xi^2}\exp\left(-\frac{\alpha\xi^2 t}{1+(1-\alpha)\xi^2}\right)\right\}\frac{\cos(y\xi)}{\xi^2}d\xi, \tag{28}$$

representing the motion induced by an infinite plate that is moving in its plane with a constant velocity or applies a constant tangential stress to the fluid. For $\alpha = 1$, as expected, these solutions attain the known forms

$$u_v(y,t) = \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right),$$
$$u_s(y,t) = \frac{2}{\pi}\int_0^\infty \frac{\cos(y\xi)}{\xi^2}\left(1-e^{-\xi^2 t}\right)d\xi. \tag{29}$$

Indeed, the first is the classical solution of Stokes' first problem and the second one is the dimensionless form of Eq. (21) from [13] where $f(t) = H(t)$. It can be also written in the simpler form as

$$u_s(y,t) = y\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right) - 2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right). \tag{30}$$

In Fig. 1a and b, for comparison, presents profiles of the dimensionless velocity fields corresponding to motions induced by an infinite plate that is moving in the plane with a constant velocity or applies a constant shear to the fluid. In each case, the fluid velocity corresponding to fractional or ordinary fluids smoothly decreases from a maximum value near the plate to an asymptotic value in response to the large values of $y$. The velocity of the fractional fluids decreases with respect to $\alpha$ and the corresponding diagrams tend to superpose over that of ordinary fluid when $\alpha \to 1$. Moreover, as it results from graphs, the fluid flows faster when the velocity is given on the boundary.

### *4.2. Case 2 (ramped-type velocity or ramped-type shear on the boundary)*

Let us now consider $f(t) = -g(t) = tH(t)$ into Eqs. (17) and (25). The obtained solutions

$$u_v(y,t)=t-\frac{2}{\alpha\pi}\int_0^\infty \frac{\sin(y\xi)}{\xi^3}\left\{1-\exp\left(-\frac{\alpha\xi^2 t}{1+(1-\alpha)\xi^2}\right)\right\}d\xi, \tag{31}$$

and

$$u_s(y,t) = -ty + \frac{2}{\pi}\int_0^\infty\left\{t-\left[1-\exp\left(-\frac{\alpha\xi^2 t}{1+(1-\alpha)\xi^2}\right)\right]\times\frac{\cos(y\xi)}{\alpha\xi^2}\right\}\frac{d\xi}{\xi^2}, \tag{32}$$

correspond to the motion generated by the plate that is constantly accelerating, respectively to the motion due to a plate that applies a ramped type [16] shear stress to the fluid.

Substituting $\alpha = 1$ into these last relations, we again recover known results from the literature, namely

$$u_v(y,t) = t-\frac{2}{\pi}\int_0^\infty\left(1-e^{-\xi^2 t}\right)\frac{\sin(y\xi)}{\xi^3}d\xi, \quad u_s(y,t) = -ty+\frac{2}{\pi}\int_0^\infty\left[t-\left(1-e^{-\xi^2 t}\right)\frac{\cos(y\xi)}{\xi^2}\right]\frac{d\xi}{\xi^2}. \tag{33}$$

Indeed, the solutions (33) represent the dimensionless forms of Eqs. (23) and (4.7) from Refs. [17], respectively [18, with $f = -1$].

Fig. 2a and b presents the velocity profiles correspond with the motions induced by a constantly accelerated plate or due to a ramped shear stress on the plate. As expected, they have the same looks as before and the diagrams corresponding to fractional fluids tend to superpose over those of ordinary fluids. The boundary conditions in the case of the motion due to the moving plate are clearly satisfied while the fluid velocities closed to the plate are more ascertainable for different values of alpha in the case of second motion.

### *4.3. Case 3 (an oscillatory velocity or an oscillating shear stress on the boundary)*

By now replacing $f(t) = -g(t) = H(t)\sin(\omega t)$ into Eqs. (17) and (25), we find the solutions corresponding to motions due to a plate that is oscillating in its plane or applies oscillatory shear stress to the fluid. Such solutions, as expected, can be expressed as sum of steady-state and transient solutions, i.e.

$$u_v(y,t) = u_{vs}(y,t) + u_{vt}(y,t), \quad u_s(y,t) = u_{ss}(y,t) + u_{st}(y,t), \tag{34}$$

where

$$u_{vs}(y,t)=\sin(\omega t)-\frac{2\alpha}{\pi}\omega\cos(\omega t)\int_0^\infty \frac{\xi\sin(y\xi)}{\alpha^2\xi^4+\omega^2\left[1+(1-\alpha)\xi^2\right]}d\xi - \frac{2}{\pi}\omega^2\sin(\omega t)\int_0^\infty \frac{\left[1+(1-\alpha)\xi^2\right]\sin(y\xi)}{\xi\{\alpha^2\xi^4+\omega^2\left[1+(1-\alpha)\xi^2\right]^2\}}d\xi, \tag{35}$$

$$u_{vt}(y,t)=\frac{2\alpha}{\pi}\omega\int_0^\infty \frac{\xi\sin(y\xi)}{\alpha^2\xi^4+\omega^2\left[1+(1-\alpha)\xi^2\right]^2}\exp\left(-\frac{\alpha\xi^2 t}{1+(1-\alpha)\xi^2}\right)d\xi, \tag{36}$$

$$u_{ss}(y,t)=-\frac{2\alpha}{\pi}\omega\cos(\omega t)\int_0^\infty \frac{\cos(y\xi)d\xi}{\alpha^2\xi^4+\omega^2\left[1+(1-\alpha)\xi^2\right]^2} + \frac{2}{\pi}\sin(\omega t)\int_0^\infty \frac{(\alpha-1)\{\alpha^2\xi^4+\omega^2\left[1+(1-\alpha)\xi^2\right]^2\}-\alpha^2\xi^2}{\left[1+(1-\alpha)\xi^2\right]\left\{\alpha^2\xi^4+\omega^2\left[1+(1-\alpha)\xi^2\right]^2\right\}}\cos(y\xi)d\xi, \tag{37}$$

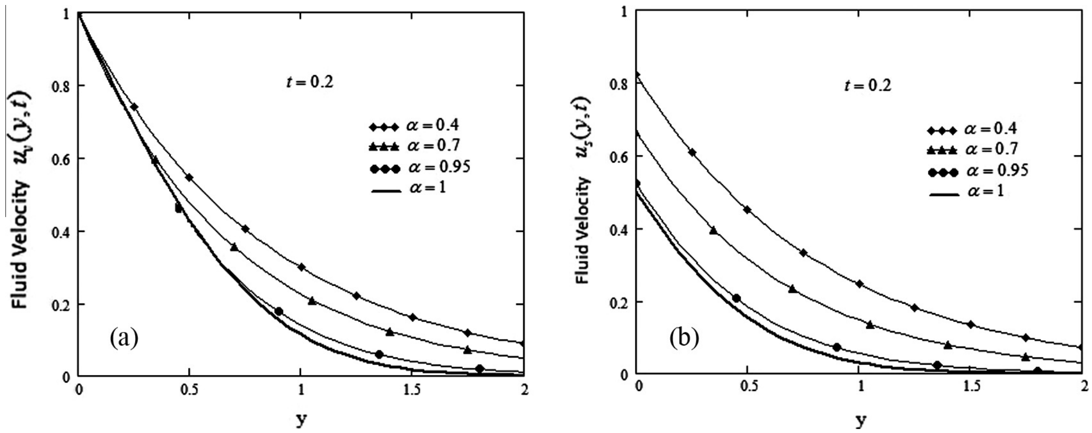


**Figure 1** Profiles of the velocities $u_v(y, t)$ and $u_s(y, t)$ given by Eqs. (27) and $(29)_1$, respectively (28) and $(29)_2$, at time $t = 0.2$ and different values of $\alpha$.

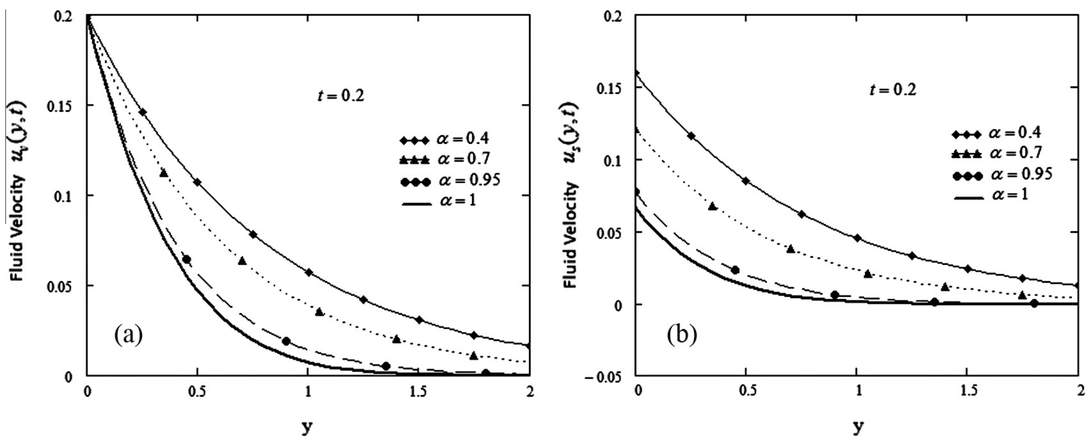


**Figure 2** Profiles of the velocities $u_v(y, t)$ and $u_s(y, t)$ given by Eqs. (31) and $(33)_1$, respectively (32) and $(33)_2$, at time $t = 0.2$ and different values of $\alpha$.

$$u_{st}(y,t)=\frac{2\alpha}{\pi}\omega\int_0^\infty \frac{\cos(y\xi)}{\alpha^2\xi^4+\omega^2\left[1+(1-\alpha)\xi^2\right]^2}\exp\left(-\frac{\alpha\xi^2 t}{1+(1-\alpha)\xi^2}\right)d\xi. \tag{38}$$

Making $\alpha = 1$ into (34), we recover the solutions

$$u_v(y,t) = u_{vs}(y,t) + u_{vt}(y,t),$$
$$u_s(y,t) = u_{ss}(y,t) + u_{st}(y,t), \tag{39}$$

$$u_{vs}(y,t) = \sin(\omega t) - \frac{2}{\pi}\omega\cos(\omega t)\int_0^\infty \frac{\xi\sin(y\xi)d\xi}{\xi^4+\omega^2} - \frac{2}{\pi}\omega^2\sin(\omega t)\int_0^\infty \frac{\sin(y\xi)}{\xi(\xi^4+\omega^2)}d\xi, \tag{40}$$

$$u_{ss}(y,t) = -\frac{2}{\pi}\omega\cos(\omega t)\int_0^\infty \frac{\cos(y\xi)d\xi}{\xi^4+\omega^2} + \frac{2}{\pi}\sin(\omega t)\int_0^\infty \frac{\xi^2\cos(y\xi)}{\xi^4+\omega^2}d\xi, \tag{41}$$

$$u_{vt}(y,t) = \frac{2\omega}{\pi}\int_0^\infty \frac{\xi\sin(y\xi)}{\xi^4+\omega^2}e^{-\xi^2 t}d\xi,$$
$$u_{st}(y,t) = \frac{2\omega}{\pi}\int_0^\infty \frac{\cos(y\xi)}{\xi^4+\omega^2}e^{-\xi^2 t}d\xi, \tag{42}$$

corresponding to the same motions of ordinary fluids. Indeed, the solution $(39)_1$ gives the starting solution obtained in [19, Eq. (4.36) with $\lambda_r = 0$] while the solution $(39)_2$ is the dimensionless form of that resulting from [20, Eq. (20)]. Of course, all steady-state solutions of this section can be processed to give simpler forms as those obtained in [19,20].

It is well known fact that the steady state solutions for oscillating motions of fluids are essential for those who want to remove the transients from their experimental work. Ultimately, in many practical situations it is vital to find the time after which the fluid flows are consistent with the steady-state solution. Fig. 3a and b clearly shows that the necessary time to attain the steady-state is an increasing function with respect to the fractional parameter $\alpha$ in the case when velocity is given on the boundary and decreases for motions owing to a shear stress on the boundary. Furthermore, this time is very small for motions due to an oscillating plate in comparison with that for motions induced by an oscillating shear stress on the boundary. The variation of this time with respect to the oscillation frequency $\omega$ is given by Fig. 4a and b. In both cases the mandatory time to attain the steady state is a decreasing function of $\omega$. It is also much lower for motions due to an oscillating plate in comparison with the second case.

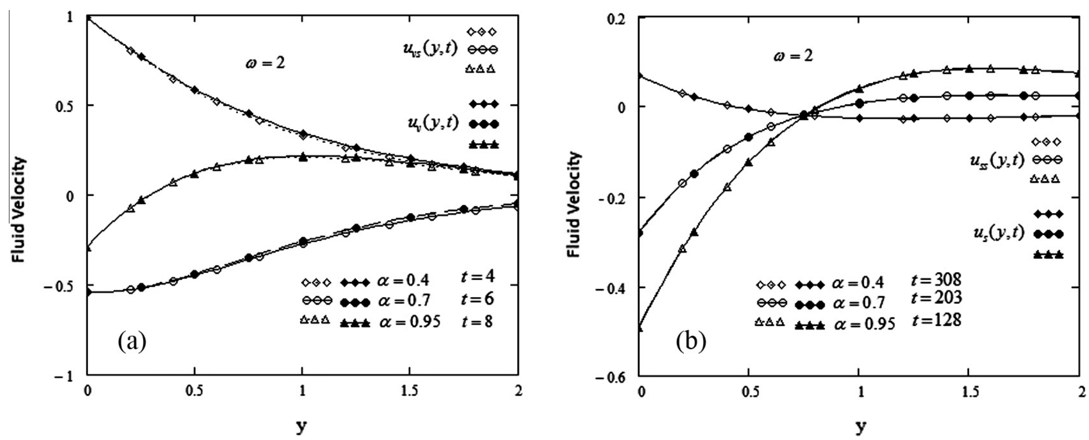


**Figure 3** The required time to reach the steady-steady for both types of motions for $\omega = 2$ and different values of $\alpha$.

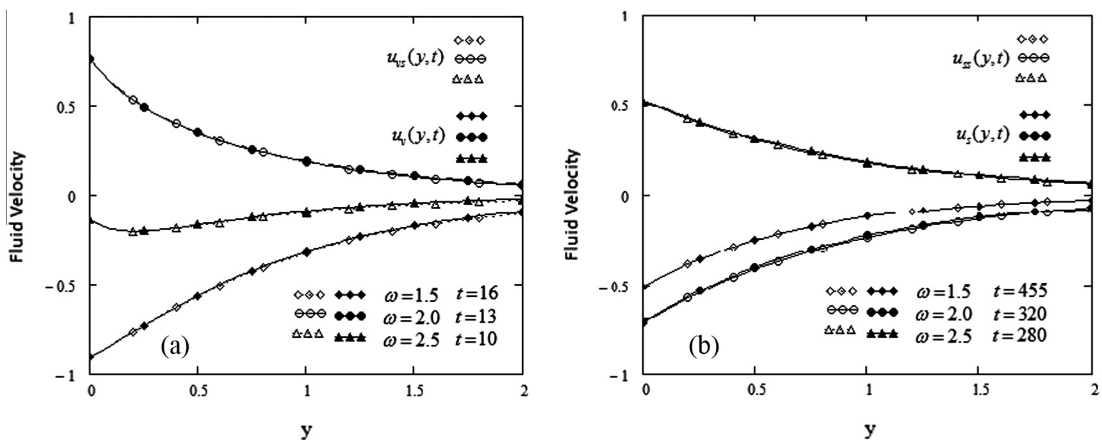


**Figure 4** Required time to reach the steady-steady for both types of motions for $\alpha = 0.3$ and different values of $\omega$.

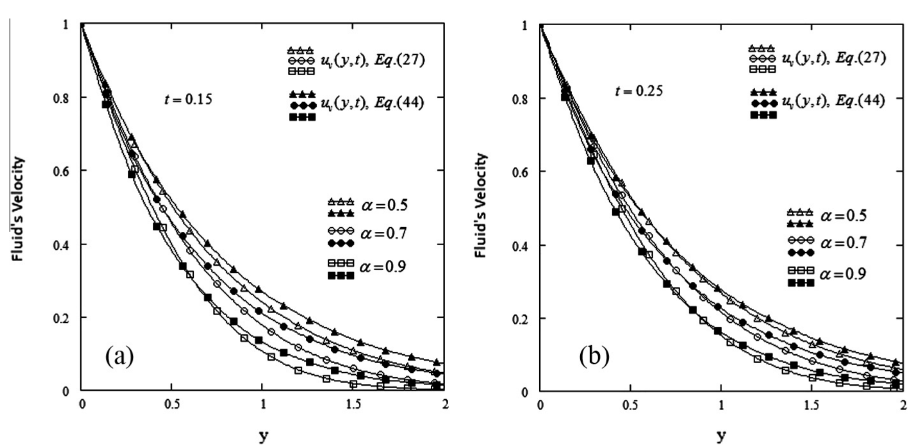


**Figure 5** Comparison of the velocities $u_v(y, t)$ obtained by, Caputo derivative and Caputo–Fabrizio derivative operators for different values of $\alpha$ at $t = 0.15$ and $t = 0.25$.

## 5. Comparison with the previous results

Finally, it is worth pointing out that the present study is not conclusive without any comparison with the previous results that have been obtained using an older definition of fractional derivative. For comparison, we shall here consider the non-dimensional form

$$u(y,t) = \frac{2}{\pi}\int_0^\infty \xi \sin(y\xi)\int_0^t s^{\alpha-1} f(t-s) E_{\alpha,\alpha}(-\xi^2 s^\alpha) ds d\xi, \quad (43)$$

of the velocity field obtained by Debnath and Bhatta [7, Eq. (6.7.51)] for the same motion of a viscous fluid but with Caputo fractional derivative [4,5]. Here, $E_{\alpha,\alpha}(z)$ is the Mittag-Leffler function and Eq. (43) reduces to our Eq. (19) for $\alpha \to 1$. Now we substitute $f(t) = H(t)$ into Eq. (43) in order to get

$$u_v(y,t) = \frac{2}{\pi}\int_0^\infty \xi \sin(y\xi) \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\xi^2 s^\alpha) ds d\xi. \quad (44)$$

The variation of the dimensionless velocity $u_v(y, t)$ given by Eqs. (27) and (44) is presented in Fig. 5 for $t = 0.15$ and 0.25 and different values of $\alpha$.

In both cases the fluid velocity smoothly decreases from a maximum value near the plate to an asymptotic value when $y \to \infty$. Near the plate the fluid with Caputo-Fabrizio fractional derivative flows slower in comparison with that with Caputo fractional derivative. An opposite trend appears further. As it is known from the existing literature [7] and is evident from our Figs. 1 and 2, both fluids flow faster in comparison with ordinary fluid.

Let us now consider the situation when the plate is oscillating in its plane. Taking $f(t) = H(t)\sin(\omega t)$ into Eq. (43) we find the starting solution

$$u_v(y,t) = \frac{2}{\pi}\int_0^\infty \xi \sin(y\xi) \int_0^t s^{\alpha-1} \sin(\omega(t-s)) E_{\alpha,\alpha}(-\xi^2 s^\alpha) ds d\xi, \quad (45)$$

which for $\alpha \to 1$ reduces to our solution (39). This solution can be also expressed as a sum of permanent and transient parts and its steady-state components is

$$u_{vs}(y,t) = \frac{2}{\pi}\int_0^\infty \xi \sin(y\xi) \int_0^\infty s^{\alpha-1} \sin(\omega(t-s)) E_{\alpha,\alpha}(-\xi^2 s^\alpha) ds d\xi. \quad (46)$$

Our interest now is to determine the approximate time to attain the steady-state corresponding to this case but our endeavor using Eqs. (45) and (46) was unfruitful, and may be due to the Mittag-Leffler function that appears in both solutions. However, in order to remove this drawback, we presented in Fig. 6 the profiles of the transient component of the starting solution (45). As it results from this figure, the required time to reach the steady-state also increases for increasing values of $\alpha$. However, it seems to be much higher in comparison with fluids with Caputo-Fabrizio fractional derivative.

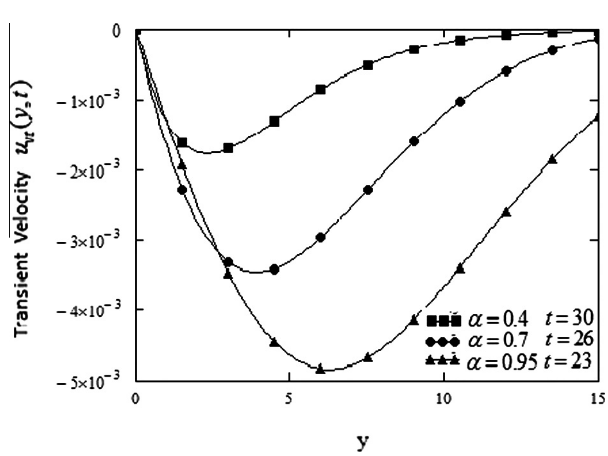


**Figure 6** Approximate time for transients to disappear from $u_v(y, t)$ with $\omega = 2$ and different values of $\alpha$.

## 6. Conclusions

In this article, exact general solutions for the dimensionless velocity and shear stress fields corresponding to the motion of a viscous fluid with fractional derivative without singular kernel over an infinite plate are established. The fluid motion is generated by plate that is moving in its plane with a velocity $Uf(t)$ or applies a shear stress $Sg(t)$ to the fluid. The exact solutions are expressed under integral form in terms of exponential and trigonometric functions and the solutions corresponding to ordinary fluids are obtained as limiting cases of these solutions when the fractional parameter $\alpha \to 1$. They satisfy all prescribed initial and boundary conditions and can be conveniently customized to obtain known results from the literature.

- Finally, three fundamental problems are taken into consideration and different graphical representations are provided and discussed.

  These are as follows:

  (a) The motion due to suddenly moved plate (Stokes first problem).
  (b) Motion induced by a constantly accelerated plate.
  (c) Motion due to an oscillating plate (Stokes second problem).

  and their corresponding motions with shear stress on the boundary. The main findings are as follows:

  (i) Fractional parameter has a significant control on the motion of the fluid and the fractional fluid flows faster in contrast to the ordinary fluid in both types, with velocity or shear stress on the boundary.
  (ii) Mandatory time to attain the steady-state for oscillating motions of fractional fluid increases with respect to the non-integer parameter $\alpha$ in the case of a motion with velocity on the boundary and decreases for motions generated by a shear stress on the boundary.
  (iii) For motions due to an oscillating plate, the steady state is later achieved for ordinary in contrast to fractional fluids. An opposite trend appears for motions generated by an oscillating shear on the boundary.
  (iv) Required time to reach the steady-state is a decreasing function with respect to the frequency $\omega$ of the oscillations. However, it is much lower for motions due to an oscillating plate in comparison with the second type of motions.
- From the comparison with the previous results that have been obtained using an older definition of fractional derivative for example Caputo fractional derivative,
  (i) For the constant velocity of the plate in both cases the fluid velocity smoothly decreases from a maximum value near the plate to an asymptotic value when $y \to \infty$.

(ii) Near the plate the fluid with Caputo-Fabrizio fractional derivative flows slower in comparison with that with Caputo fractional derivative, an opposite trend appears further.

(iii) For the case when the plate is oscillating in its plane, the required time to reach the steady-state also increases for increasing values of fractional order parameter. However, it seems to be much higher for the model with Caputo fractional derivative in comparison with fluid model with Caputo-Fabrizio fractional derivative.


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